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Conditional Probabilities with a Quantal and a Kolmogorovian Limit

Sven Aerts[1]

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We give a definition for the conditional probability that is applicable to quantum situations as well as classical ones. We show that the application of this definition to a two-dimensional probabilistic model, known as the epsilon model, allows one to evolve continuously from the quantum mechanical probabilities to the classical ones. Between the classical and the quantum mechanical, we identify a region that is neither classical nor quantum mechanical, thus emphasizing the need for a probabilistic theory that allows for a broader spectrum of probabilities.

1. INTRODUCTION

In this article we want to present a two-dimensional (i.e., only two outcomes are related to each measurement) probabilistic model that allows for a quantum mechanical as well as a classical description. When we say classical or quantum mechanical we mean that the *probabilities* related to a measurement are the same as those that are computed by these respective theories. Thus, by the term probability itself (as contrasted to a CSD, which we will introduce shortly) we denote the limit of the relative frequencies (von Mises, 1919) and not the Kolmogorovian sense (unless explicitly stated) of probability with all its mathematical connotations. In order to talk more easily about the two situations, we will introduce the following abbreviations:

- We will call a quantum statistical description (QSD) any description of the outcomes related to an experiment that gives rise to a set of

[1]TENA, Free University of Brussels, Pleinlaan 2, 1050 Brussels, Belgium; e-mail: saerts@vub.ac.be.

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probabilities that are derivable within the context of Hilbert-space quantum mechanics.

- We will call a classical statistical description (CSD) any description of the outcomes related to an experiment that gives rise to a set of probabilities that are derivable within the context of the standard theory of probabilities.

Historically, the foundations of the latter were due mainly to Cardano and Laplace, but whenever we talk about CSD we will refer to the Kolmogorovian theory of probability, because of its mathematical completeness. As is well known, the Kolmogorovian approach is based on a probability space that is characterized by a triple (Ω, Σ, μ), where Ω is a nonempty set, Σ the σ-algebra of subsets of Ω, and μ a probability measure on Σ satisfying several conditions such as additivity. The reader may feel the need for a criterion that allows one to decide whether probabilities are derivable within a classical or a quantum context. Such criteria exist in the literature: they are called statistical polytopes. The most famous among these polytopes is the Bell inequality. However, since the Bell polytope relates to a correlation probability and we are going to discuss transition and conditional probabilities, we will not use this particular polytope.

As pointed out by Accardi and Fedullo (1982), for the transition probabilities related to a two-dimensional measurement (that is, a measurement with two possible outcomes) it is sufficient to consider three (as is well known, Bell uses the probabilities related to four different measurements on a singlet state) transition probabilities to see whether they belong to a QSD or to a CSD. They also show that for the two-dimensional case the set of probabilities belonging to a CSD form a subset of the set of probabilities belonging to a QSD. Gudder (1984) derived similar results (albeit through a different approach) for the case of a CSD. We will use his second theorem, which we state without proof.

Theorem 1 (Gudder, 1984). Let $0 \leq p_1, p_2, p_3 \leq 1$. Then there exist events A, B, C in a probability space (Ω, Σ, μ) such that $\mu(A) = \mu(B) = \mu(C) = 1/2$ and $\mu(A \cap B) = p_1$, $\mu(A \cap C) = p_2$, and $\mu(B^c \cap C) = p_3$ iff $p_i \leq p_j + p_k$, $i \neq j \neq k = 1, 2, 3$, and $p_1 + p_2 + p_3 \leq 1$.

One can easily see that not every triple p_1, p_2, p_3 will comply with the restrictions as expressed by the inequalities in this theorem. Those that will comply can be derived by the use of the orthodox framework of Kolmogorovian probability theory or, to express it in our language, permit a CSD. Besides Bell and Accardi and Gudder, we also mention the nice monograph on these restrictions known as polytopes by Pitovsky (1989).

2. A MACROSCOPIC DEVICE PRODUCING A QUANTUM PROBABILISTIC STRUCTURE

We would like to expose a very simple macroscopic device that has a QSD. The possibility of building such a device became clear at least as early as 1980, when Dirk Aerts constructed a model that could be built by any plumber (consisting of two communicating vessels and two siphons to perform the measurement) and that violates the Bell inequalities (the value the polytope takes is four, which is the largest violation possible). The model we are going to present in this article was found by Dirk Aerts in 1989, and has been one of the inspirations of our Brussels group. We want to emphasize that this model has been introduced in several previous articles and that we only briefly review it here so that this article is self-contained.

The model consists of a physical entity S that is a point particle P that can move on the surface of a sphere, denoted *sur*, with center O and radius 1. The unit vector v where the particle is located on *sur* represents the state p_v of the particle (Fig. 1). For each point $u \in sur$ we introduce the following experiment e_u. We consider the diametrically opposite point $-u$ and install

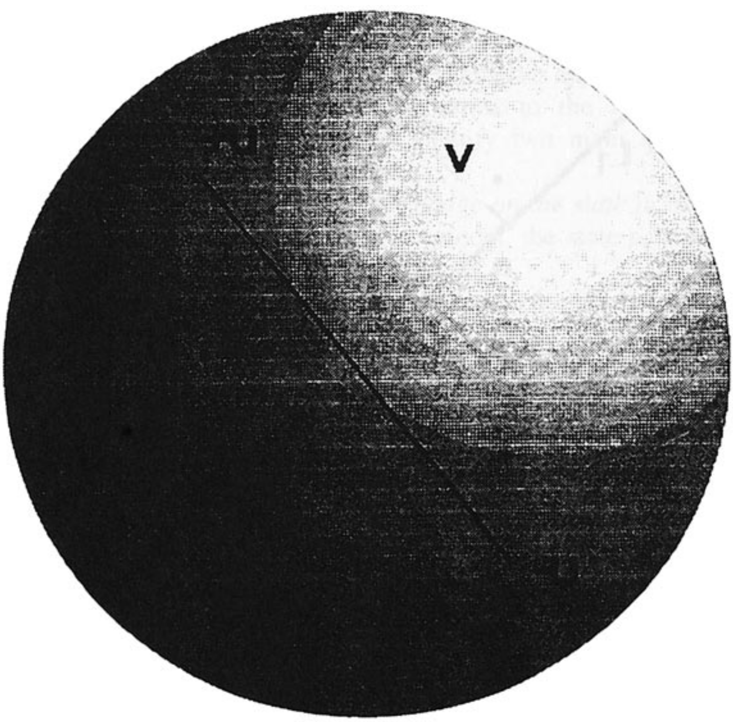

Fig. 1. The sphere model; **u** represents the measurement direction, **v** the entity.

a piece of elastic of length 2 such that it is fixed with one of its endpoints in u and the other endpoint in $-u$.

Once the elastic is installed, the particle P falls from its original place v orthogonally onto the elastic and sticks on it. Then the elastic breaks and the particle P attached to one of the two pieces of the elastic moves to one of the two endpoints u or $-u$. Depending on whether the particle P arrives at u or $-u$, we give the outcome o_1^u or o_2^u to e_u. In Fig. 2 we represent the disk of the sphere where the experiment e_u takes place, and we can easily calculate the probabilities corresponding to the two possible outcomes.

Therefore we remark that the particle P arrives at u when the elastic breaks at a point of the interval L_1 and arrives at $-u$ when it breaks at a point of the interval L_2 (Fig. 2). We make the hypothesis that the elastic breaks uniformly, which means that the probability that the particle being in state p_v arrives at u is given by the length of L_1 (which is $1 + \cos\theta$) divided by the length of the total elastic (which is 2). The probability that the particle in state p_v arrives at $-u$ is the length of L_2 (which is $1 - \cos\theta$) divided by

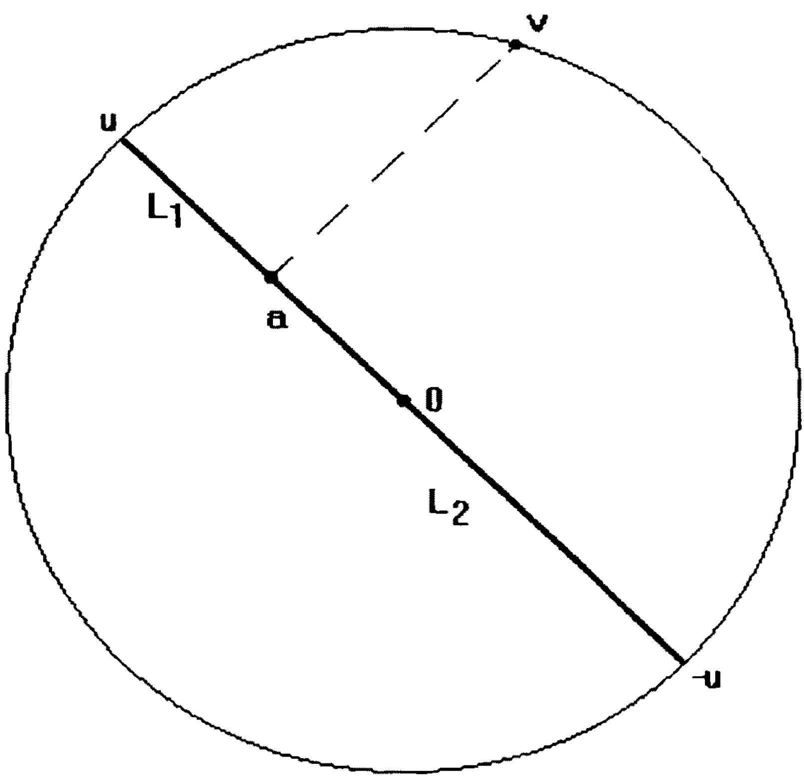

Fig. 2. The intersection of the sphere with the plane $(u, 0, v)$.

the length of the total elastic. If we denote these probabilities respectively by $P(o_1^u, p_v)$ and $P(o_2^u, p_v)$, we have

$$P(o_1^u, p_v) = \frac{1 + \cos\theta}{2} = \cos^2\frac{\theta}{2} \tag{1}$$

$$P(o_2^u, p_v) = \frac{1 - \cos\theta}{2} = \sin^2\frac{\theta}{2} \tag{2}$$

These transition probabilities are the same as the ones related to the outcomes of a Stern–Gerlach spin experiment on a spin-1/2 quantum particle of which the quantum-spin state in direction $v = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$, denoted by $\bar{\psi}_v$, and the experiment e_u corresponding to the spin experiment in direction $u = (\cos\beta \sin\alpha, \sin\beta \sin\alpha, \cos\alpha)$, is described respectively by the vector and the self-adjoint operator

$$\psi_v = (e^{-i\phi/2}\cos\theta/2, e^{i\phi/2}\sin\theta/2), \qquad H_u = \frac{1}{2}\begin{pmatrix} \cos\alpha & e^{-i\beta}\sin\alpha \\ e^{i\beta}\sin\alpha & -\cos\alpha \end{pmatrix}$$

of a two-dimensional complex Hilbert space.

As noted on many previous occasions, the probabilities as they appear in this example are due to *a lack of knowledge about the measurement process*. More precisely, we could say that we lack the knowledge of where exactly the elastic breaks during a measurement. Therefore this approach was baptized the hidden measurement approach in contrast to the hidden variables approaches. More specifically, we can identify two main aspects of the experiment e_u as it appears in the model.

1. The experiment e_u effects a *real change on the state* p_v of the entity S. As explained in the introduction of the model, the state p_v changes into one of the states p_u or p_{-u} by the experiment e_u.

2. The probabilities appearing are due to a *lack of knowledge about a deeper reality of the individual measurement process* itself, namely where the elastic breaks.

One can show that these two effects give rise in general to quantumlike structures. One can extend these ideas to the n-dimensional case (D. Aerts, 1986) or even to the countably infinite-dimensional case (Coecke, 1996). One can think of other fields of scientific interest where a model that incorporates an intrinsic influence of the measurer on the measured would be valuable, such as psychology (D. Aerts and S. Aerts, 1995). For another relevant discussion, see Czachor (1992).

3. THE CLASSICAL LIMIT

A most interesting consequence of the sphere model is that one can limit the magnitude of the lack of knowledge about the measurement situation.

D. Aerts *et al.* (1993) studied the sphere model in the context of lattice theory under varying "lack of knowledge," parametrizing this variation by a number $\epsilon \in [0, 1]$ such that $\epsilon = 1$ corresponds to the situation of maximal lack of knowledge, giving rise to a quantum structure, and $\epsilon = 0$ corresponds to the situation of zero lack of knowledge, generating a classical structure, and other values of ϵ correspond to intermediate situations, giving rise to a structure that is neither quantum nor classical. This model was called the ϵ-model. We want to make an analogous study, but in the context of probabilities rather than lattices. Let us show how the ϵ-model works. We start from the sphere model, but introduce different types of elastics. An ϵ-elastic consists of three different parts: one lower part where it is unbreakable, a middle part where it breaks uniformly, and an upper part where it is again unbreakable. By means of the parameter ϵ in [0, 1], we fix the sizes of the three parts in the following way. Suppose that we have installed the ϵ-elastic between the points $-u$ and u of the sphere. Then the elastic is unbreakable in the lower part from $-u$ to $-\epsilon \cdot u$, it breaks uniformly in the part from $-\epsilon \cdot u$ to $\epsilon \cdot u$, and it is again unbreakable in the upper part from $\epsilon \cdot u$ to u (Fig. 3).

An e_u experiment performed by means of an ϵ-elastic shall be denoted by e_u^ϵ.

We have the following cases:

1. $v \cdot u \leq -\epsilon$. For a fixed p_v let us define

$$eig(\{o_2^u\}) = \{u \mid v \cdot u \leq -\epsilon\} \tag{3a}$$

The particle sticks to the lower part of the ϵ-elastic, and *any* breaking of the elastic will pull it down to the point $-u$. We have $P^\epsilon(o_1^u, p_v) = 0$ and $P^\epsilon(o_2^u, p_v) = 1$. This explains why we have abbreviated the above-defined set $eig(\{o_2^u\})$, i.e., they form a deterministic subset of the set of all states.

2. $-\epsilon < v \cdot u < \epsilon$. Similar to the previous case, we define

$$sup(\{o^u\}) = \{u \mid -\epsilon < v \cdot u < \epsilon\} \tag{3b}$$

The particle falls onto the breakable part of the ϵ-elastic. We can easily calculate the transition probabilities and find

$$P^\epsilon(o_1^u, p_v) = \frac{1}{2\epsilon}(v \cdot u + \epsilon) \tag{4a}$$

$$P^\epsilon(o_2^u, p_v) = \frac{1}{2\epsilon}(\epsilon - v \cdot u) \tag{4b}$$

We see that for all $u \in sup(\{o^u\})$ the outcome is uncertain. That is why we have called this set metaphorically the superpositionset of u.

3. $\epsilon \leq v \cdot u$. As above, we define

$$eig(\{o_1^u\}) = \{u \mid \epsilon \leq v \cdot u\} \tag{3c}$$

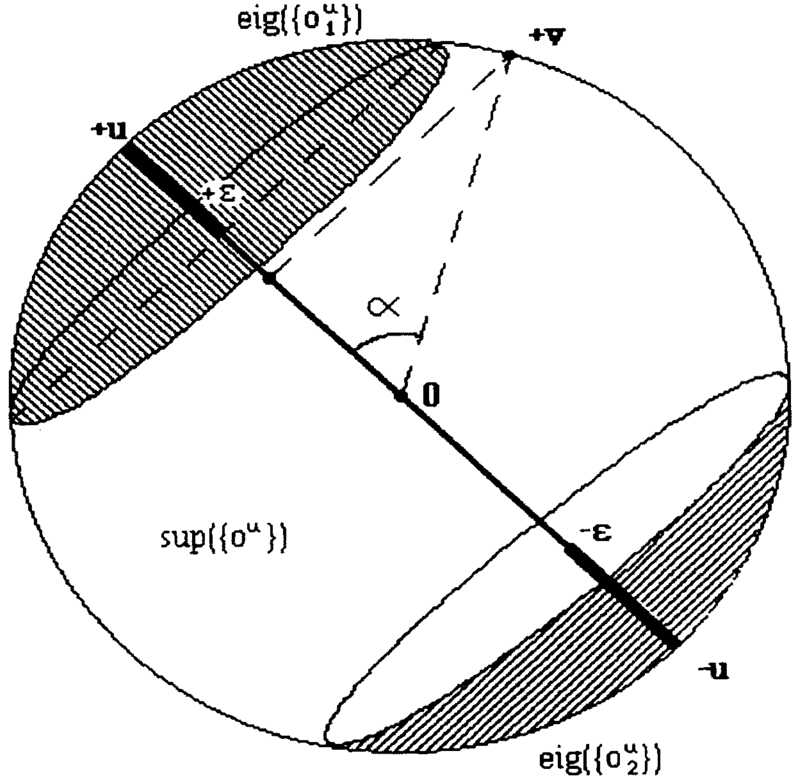

Fig. 3. The epsilon sphere with its different regions.

Then the particle falls onto the upper part of the ϵ-elastic, and any breaking of the elastic will pull it upward such that it arrives at u. We have $P^\epsilon(o_1^u, p_v) = 1$ and $P^\epsilon(o_2^u, p_v) = 0$.

3.1. Deterministic Is Not Yet Classical

We have indicated that the limit $\epsilon \to 0$ is equivalent to reducing the lack of knowledge on the interaction between the measurer and the measured. We have also indicated that this interaction is one of the sources of the nonclassical behavior of this model. Can we conclude now that this model has a CSD if we set $\epsilon = 0$? Let us test this hypothesis with the polytope Gudder provided which we stated in the first section of this article. According to the theorem we need to find three numbers that violate the inequalities in order to show this model will not yet allow a CSD. Let us show this explicitly.

Theorem 2. The transition probabilities of the ϵ-model with $\epsilon = 0$ will not allow a description in terms of a probability triple.

Let us call A, B, and C the events that the entity will produce the result "up" when it is measured as being in a state **u**, **v**, and **w**, respectively. We can easily see that $\mu(A) = \mu(B) = \mu(C) = 1/2$ for any **u**, **v**, and **w**. Let us give these three vectors an explicit geometrical representation in order to calculate $\mu(A \cap B) = p_1$, $\mu(A \cap C) = p_2$, $\mu(B^c \cap C) = p_3$. Let us choose **u**, **v**, and **w** such that they are all contained in one plane, and that **v** is obtained by rotating **u** over an angle of $2\pi/3$ and **w** by rotating **v** over the same angle (this means that there is also an angle $2\pi/3$ between **w** and **u**). Now for the case $\epsilon = 0$ the probability has become a Heaviside step function and we can easily see that, since the angle for p_1 and p_2 is greater than $\pi/2$, the probabilities p_1 and p_2 both equal zero. However, for p_3 we have to take the angle between $-\mathbf{v}$ and **w**, which is $\pi/3 < \pi/2$. Therefore $p_3 = 1$. We see that indeed $p_1 + p_2 + p_3 \leq 1$, but that $p_3 > p_1 + p_2$. We conclude that no such three events can be formulated in a single probability triple. For a more general proof of this kind, see T. Durt (1996).

Gudder (1984) proposes a more general scheme for probability triples in order to see if it is possible to fit quantum mechanics in such a generalized theory. We will take another approach, since we would like to construct a model that has a Kolmogorovian probability limit. Therefore we will look at families of conditional probabilities.

3.2. The Concept of Conditional Probability

While we can deal with transition probabilities in the domain of quantum mechanics, there is no such thing in the classical regime. The probability concept that provides a bridge between the classical and quantum regimes is the conditional probability. Normally the conditional probability is introduced by means of the Bayes axiom. As is well known (for example, Gudder, 1988) there is a serious drawback when trying to use the Bayes axiom in quantum mechanics: it is a nonoperational definition for noncompatible observables, because the observables do not take their values simultaneously. This is often stated as the problem of the nonexistence of a joint-probability distribution (see, however, Cohen, 1986). Still it will be clear that the preparation of an entity is in fact a kind of conditioning for any consecutive measurement. The point is that there is a distinction between the occurrence of an outcome when an experiment is actually performed and the conditioning (preparation) on an outcome corresponding to an experiment. Following D. Aerts (1995), we will propose a natural extension of the concept of conditional probability that is operational both in the quantum and in the classical regime, but first we want to give a precise definition of the conditioning:

Definition 1. Given a situation μ of lack of knowledge on the states of an entity S described by a probability measure on this set of states Σ, we condition the entity S on a subset $A_f \subset O_f$ (the set of possible outcomes related to f) for an experiment f if we consider during the performance of the experiment e only those trials where the situation of the entity before the experiment e is such that we can predict the outcome for the experiment f to occur with certainty in A_f if we would decide to perform the experiment f.

In a less precise language one could say that conditioning is equivalent to a change of the situation μ before the experiment in such a way that the experiment f would give with certainty an outcome in A_f *if it would be executed.* The new situation of lack of knowledge is described by the probability measure that we shall denote by μ_{A_f}: $\mathcal{B}(\Sigma) \to [0, 1]$. It is defined as follows; for an arbitrary subset $K \subset \Sigma$

$$\mu_{A_f}(K) = \mu(K \cap eig(A_f))/\mu(eig(A_f)) \tag{5}$$

Now that we have introduced this concept of "conditioning" on an experiment, we can introduce the general concept of conditional probability.

Definition 2. Given a situation μ of lack of knowledge on the states of an entity described by the probability measure μ, and given two experiments e and f, then we want to consider the conditional probability $P(A_e, A_f, \mu)$ that the experiment e makes occur an outcome in the set A_e when the situation is conditioned on the set A_f for the experiment f. The conditional probability is a map P: $\mathcal{B}(O_e) \times \mathcal{B}(O_f) \times \mathcal{M}(\Sigma) \to [0, 1]$.

One can see how this definition reduces to the Bayes axiom if there are no state transitions, because there is no longer any difference between a possible and an actual performed measurement (see Definition 1). Also one can see how this probability is the one that is measured in the laboratory of an experimental quantum physicist (because the preparation of a state is in fact nothing more than stating that if we would repeat the experiment, we would have the same outcome with probability one).

We suggest to calculate this operational definition of conditional probability on the ϵ-model and analyze the probabilistic structure that emerges. In particular we want to show how the conditional probability on the ϵ-model evolves continuously from the quantum transition probability for the case of $\epsilon = 1$ to a classical Kolmogorovian probability satisfying Bayes' formula for the case $\epsilon = 0$. We shall also point out that for a region of intermediate values of ϵ the conditional probability is neither the quantum transition probability nor the classical Bayesian conditional probability.

3.3. The Conditional Probability and the ϵ-Model

Given a situation μ of lack of knowledge about the state of the point particle described by a uniform probability measure on the sphere corresponds to the situation where the particle P is distributed at random on the sphere. For a fixed ϵ, there are also given the two experiments e_u^ϵ and e_w^ϵ. In general we consider the conditional probability for arbitrary elements in the set of measurable subsets of the outcome sets for the two experiments. Since to calculate the conditional probability in the ϵ-model we only need one experiment e_u^ϵ with conditioning on w and with two outcomes o_1^u, o_2^u, we hope the reader will forgive us for changing the heavy notation for a lighter one. We shall denote the conditional probability that the experiment e_u^ϵ gives the outcome o_1^u (respectively o_2^u) when the entity is conditioned for the outcome o_1^w of the experiment e_w^ϵ by $p(o_1^u | o_1^w) = p(\epsilon, u, w)$, respectively $p(\epsilon, -u, w)$. We have, of course,

$$p(u_1^u | o_1^w) + p(o_2^u | o_1^w) = p(\epsilon, u, w) + p(\epsilon, -u, w) = 1$$

Thus the conditional probability $p(\epsilon, u, w)$ is the probability that the experiment e_u^ϵ gives the outcome o_1^u if the entity is conditioned on the outcome o_1^w for the experiment e_w^ϵ. This means that the lack of knowledge on the states is such that if we would decide to perform the experiment e_w^ϵ, the outcome o_1^w would come out with certainty. In other words (see Fig. 4), the state of the entity is such that the probability of the particle being in the spherical sector is distributed uniformly inside the spherical sector $eig(\{o_1^w\})$, the gray area in Fig. 4. The explicit calculation of this conditional probability can be seen to become a surface integral with the transition probability as the integrand, but now with w as a function of the infinitesimal surface element.

The actual calculation of the integral is complicated by the fact that one has to use a different integrand for different regions. Indeed, as can be seen

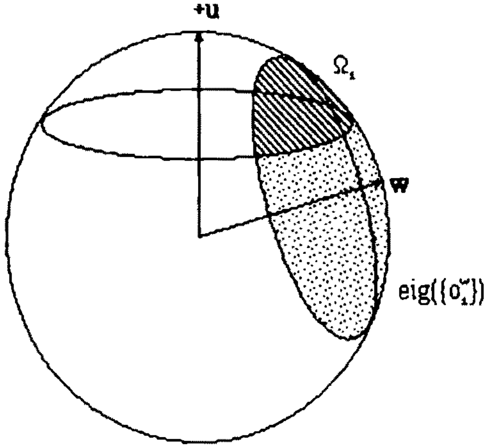

Fig. 4. The dotted part represents all possible states that correspond to our conditioning. The shaded region represents those states that will evolve deterministically toward $+u$ when measured in that direction. We denote this region $\Omega_1 = eig(\{o_1^w\}) \cap eig(\{o_2^u\})$.

from Fig. 5a, the region where $eig(\{o_1^w\})$ and $eig(\{o_1^u\})$ intersect contributes in a deterministic way to the probability in the sense that if the entity would be in this region it would always give an outcome o_1^u. Likewise, if the entity would be in the region $eig(\{o_1^w\}) \cap eig(\{o_2^u\})$ it would never contribute to the outcome o_1^u. This means that for this first region the integrand becomes 1, while for the second region it becomes 0.

Let us introduce the following abbreviations (see Fig. 5a):

$$\Omega_0 = eig(\{o_2^u\}) \cap eig(\{o_1^w\}) \tag{6a}$$

$$\Omega_s = sup(\{o^u\}) \cap eig(\{o_1^w\}) \tag{6b}$$

$$\Omega_1 = eig(\{o_1^u\}) \cap eig(\{o_1^w\}) \tag{6c}$$

$$\Omega_{\text{tot}} = \Omega_0 \cup \Omega_s \cup \Omega_1 = eig(\{o_1^w\}) \tag{6d}$$

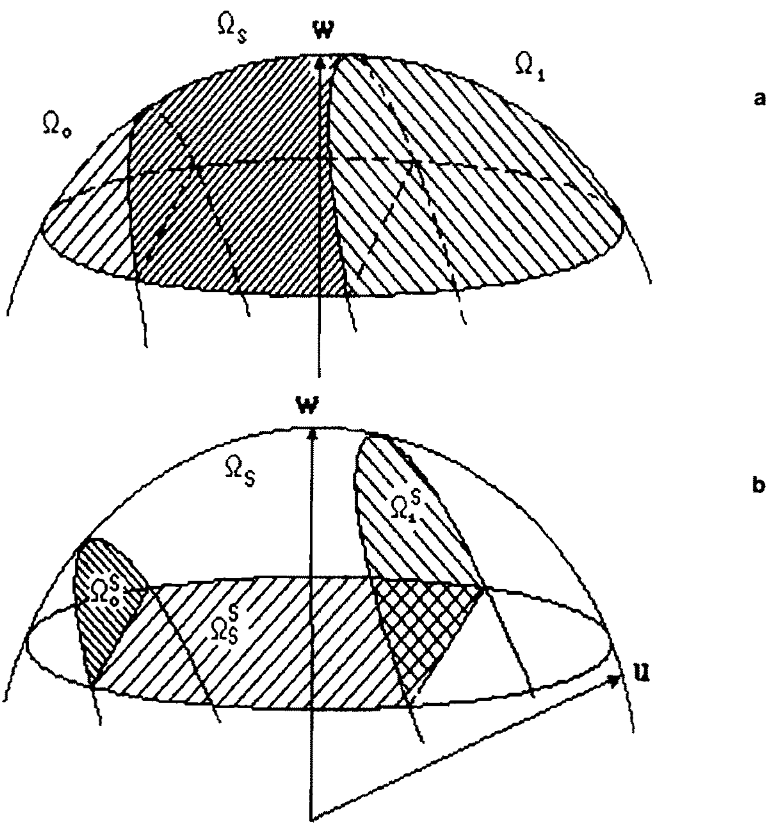

Fig. 5. (a) The different integration regions. (b) The different segments that close Ω_S.

One can very easily calculate when these spherical sectors we have identified as Ω_0, Ω_s, and Ω_1 come into action. One can indeed see that Ω_0 is not empty as soon as $\epsilon < \sin(\alpha/2)$. Likewise we find that Ω_1 contributes to the integral as soon as $\epsilon > \cos(\alpha/2)$.

The superposition region that we have called Ω_s will contribute as long as $\sin(\alpha/2) < \epsilon < \cos(\alpha/2)$.

The actual integral that needs to be calculated can be written as

$$I = \iint_{\Omega_0} 0 \cdot d\Omega_0 + \iint_{\Omega_s} \frac{\mathbf{u} \cdot \mathbf{w} + \epsilon}{2\epsilon} \, d\Omega_s + \iint_{\Omega_1} 1 \cdot d\Omega_1 \tag{7}$$

The conditional probability is proportional to this integral, the constant of proportionality being the normalization factor. This is easily found to be equal to the surface area of the spherical sector around w that we have called $eig(\{o_1^w\}) = \Omega_{\text{tot}}$.

Let us denote by Ω_0, Ω_1, and Ω_s not only the sets themselves, but also whenever they appear "plainly" (that is, not as a differential or as a boundary) as the respective surface areas belonging to these sets. It is easy to see that $\Omega_{\text{tot}} = 2\pi(1 - \epsilon)$.

Putting this together, we find that the quantity we need to calculate is

$$p(\epsilon, u, w) = \frac{1}{\Omega_{\text{tot}}} \left(\Omega_1 + \iint_{\Omega_s} \frac{\mathbf{u} \cdot \mathbf{w} + \epsilon}{2\epsilon} \, d\Omega_s \right)$$

or

$$p(\epsilon, u, w) = \frac{1}{\Omega_{\text{tot}}} \left(\Omega_1 + \frac{\Omega_s}{2} + \frac{1}{2\epsilon} \iint \mathbf{u} \cdot \mathbf{w} \, d\Omega_s \right) \tag{8}$$

Since $\mathbf{w}$ is always perpendicular to the surface of the sphere and has norm 1, it can be considered as a surface normal:

$$\mathbf{w} \, d\Omega_s = d\boldsymbol{\Omega}_s$$

By use of Gauss' theorem, we may rewrite the integral to contain only surfaces that are related to the sphere, which eliminates the difficult problem of integrating the scalar product. However, in order to apply Gauss' theorem, we need to have a closed surface and thus we need to close Ω_s by arbitrary surfaces. If we take the three segments of the circles that close the spherical caps (see Fig. 5b) and call the surfaces related to these two segments Ω_0^s, Ω_s^s, and Ω_1^s, we can write

$$\Omega_{\text{closed}} = \Omega_s + \Omega_0^s + \Omega_1^s$$

For this Ω_{closed} we may apply Gauss' theorem:

$$\iint_{\Omega_{\text{closed}}} \mathbf{u}\cdot d\mathbf{\Omega}_{\text{closed}} = \iiint \nabla\cdot\mathbf{u}\, dV$$

Since $\mathbf{u}$ is a constant vector, $\nabla\cdot\mathbf{u}$ becomes zero. Therefore we have

$$\iint_{\Omega_s} \mathbf{u}\cdot d\mathbf{\Omega}_s + \iint_{\Omega_0^s} \mathbf{u}\cdot d\mathbf{\Omega}_0^s + \iint_{\Omega_1^s} \mathbf{u}\cdot d\mathbf{\Omega}_1^s + \iint_{\Omega_s^s} \mathbf{u}\cdot d\mathbf{\Omega}_s^s = 0$$

Let α be the angle between the two vectors u and w. We can easily see (Fig. 5b) that, since the normal to the surface Ω_1^s (resp. Ω_0^s) is parallel (resp. antiparallel) to $\mathbf{u}$,

$$\iint_{\Omega_s} \mathbf{u}\cdot d\mathbf{\Omega}_s = \cos(\alpha)\cdot\Omega_s^s + \Omega_0^s - \Omega_1^s \tag{9}$$

Using (6d), (8), and (9), we are finally able to express the conditional probability in terms that relate only to surfaces on and in the sphere:

$$p(\epsilon, u, w) = \frac{1}{2\Omega_{\text{tot}}}[\Omega_{\text{tot}} + \Omega_1 - \Omega_0 + \epsilon^{-1}(\cos(\alpha)\Omega_s^s + \Omega_0^s - \Omega_1^s)] \tag{10}$$

We present a graph of this function in Fig. 6. An explicit function of only ϵ and α is easily obtained. Since the result is rather long (see the Appendix) and does not contribute to an understanding of the way the two limits arise, we shall derive the classical and quantum mechanical limits ($\epsilon \to 0$ and $\epsilon \to 1$) directly on this last result.

3.3.1. The Classical Case ($\epsilon = 0$)

If $\epsilon \to 0$ we can see that there are no longer any superposition zones. Therefore we have

$$\Omega_s = \Omega_s^s = 0$$

We can also see that $\Omega_0^s = \Omega_1^s$. The conditional probability (10) becomes

$$p(\epsilon = 0, u, w) = \frac{1}{2\Omega_{\text{tot}}}[\Omega_{\text{tot}} + \Omega_1 - \Omega_0] = \frac{\Omega_1}{\Omega_{\text{tot}}} \tag{11}$$

Now suppose one is asked what the probability is that the entity would be found in the upper half of the sphere ($eig(\{o_1^u\})$) when we know for certain that it resides in $eig(\{o_1^w\}) = \Omega_{\text{tot}}$. If one would apply Bayes' axiom with a

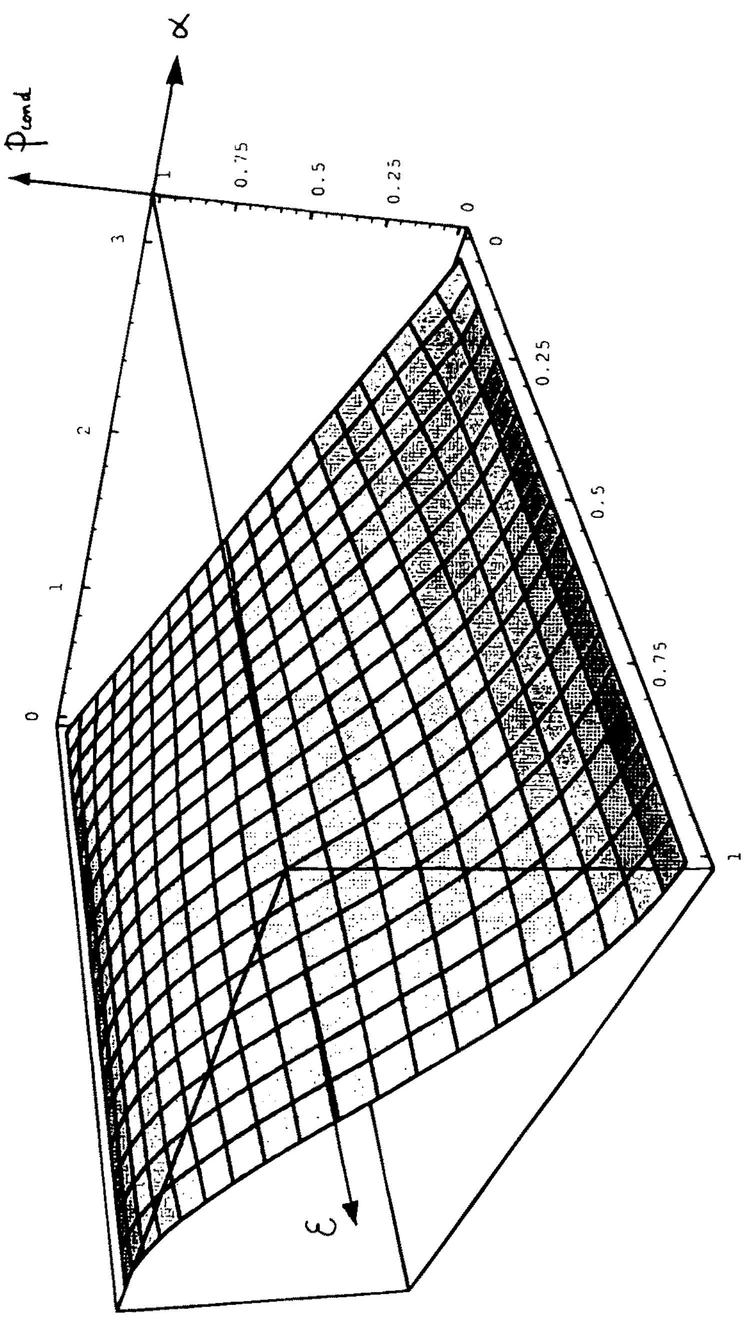

Fig. 6. The conditional probability. One can clearly see how the probability evolves continuously from the quantum to the Bayesian probability.

uniform probability measure μ, using (6c) and (6d), one would come up with the following result:

$$\frac{\mu(eig(\{o_1^u\}) \cap eig(\{o_1^w\}))}{\mu(eig(\{o_1^w\}))} = \frac{\Omega_1}{\Omega_{tot}} \tag{12}$$

This is exactly (11), which we found for the $\epsilon \to 0$ case.

3.3.2. *The Quantum Case ($\epsilon = 1$)*

In this case,

$$\Omega_1 = \Omega_0 = \Omega_1^s = \Omega_0^s = 0$$

The conditional probability (10) then becomes

$$p(\epsilon = 1, u, w) = \frac{1}{2\Omega_{tot}} [\Omega_{tot} + \cos(\alpha)\Omega_s^s]$$

But if $\epsilon \to 1$ we find that $\Omega_{tot} = \Omega_s = \Omega_s^s$:

$$p(\epsilon = 1, u, w) = \frac{1}{2\Omega_{tot}} [\Omega_{tot} + \Omega_{tot} \cos(\alpha)] = \cos^2\left(\frac{\alpha}{2}\right) \tag{13}$$

which is the well-known quantum transition probability (1) between the states p_u and p_w.

Of course, these limits may also be obtained by use of the explicit expression from the Appendix. If one does so, one finds that the limits are well defined and one arrives at the same results. A look at the graph in Fig. 6 shows that indeed the conditional probability $p(\epsilon, u, w)$ evolves continuously from the quantum transition probability between the states p_u and p_w to a linear function of the angle between the two vectors u and w which satisfies the Bayes axiom.

3.3.3. *An Intermediate Situation of the ϵ-Model That Is Neither Classical nor Quantum*

For a certain interval of intermediate values of ϵ the conditional probabilities of the ϵ-model cannot be fitted into a quantum probability model or into a classical probability model. As pointed out in the introduction, Accardi and Fedullo (1982) proved that for a two-dimensional measurement the set of possible probabilities in a CSD constitute a subset of the possible probabilities of a QSD. Therefore it is sufficient to show that for an interval of intermediate ϵ values the set of probabilities related to the outcomes of three measurements does not fit in a two-dimensional Hilbert-space description.

This we have done elsewhere (D. Aerts and S. Aerts, 1995) and we will not repeat the proof here.

4. CONCLUSIONS

We have briefly repeated that it is possible to recover the quantum mechanical probabilities if one assumes that there is a lack of knowledge about the measurement situation rather than about the entity under observation. This lack of knowledge can be parametrized. Together with the notion of conditional probability this leads to a model that incorporates a quantum and a classical statistical description. This result is intriguing because the mathematical theories that describe quantum and classical phenomena (i.e., Hilbert-space quantum mechanics and Kolmogorovian probability theory) are very distinct and it is not clear how one could proceed to form a bridge between them. What is somewhat disappointing is the fact that all results are derived for measurements with only two possible outcomes. We believe the essential content of the results (the classical and quantum mechanical limits) can be carried over to measurements with more outcomes, although the actual calculation could prove difficult. Lastly we point out that in the intermediate region the probabilities of the model are not contained in a classical or in a quantum mechanical framework, challenging the commonly held view that a set of probabilities needs to fit in either one of the descriptions.

APPENDIX

An explicit representation of the conditional probability as a function of α and ϵ is easily obtained from (10) because the surfaces that appear in the formula for the conditional probability are related to simple circle segments (S. Aerts, 1994):

$$p(\alpha, \epsilon) = p_1 H\left(\epsilon - \cos\frac{\alpha}{2}\right) + H\left(\epsilon - \sin\frac{\alpha}{2}\right) p_2 H\left(\cos\frac{\alpha}{2} - \epsilon\right) + p_3 H\left(\sin\frac{\alpha}{2} - \epsilon\right)$$

where $H(x)$ is the Heaviside function and

$$p_1 = \frac{\cos\alpha(1+\epsilon)}{4\epsilon} + \frac{1}{2}$$

$$p_2 = p_1 + \frac{1}{2} + \frac{\omega(u, w)}{4\pi(1-\epsilon)} + \frac{\cos\alpha + 1}{4\pi\epsilon(1-\epsilon)}\sigma(u, w)$$

$$p_3 = p_1 + \frac{\omega(u, w) - \omega(-u, w)}{4\pi(1-\epsilon)} + \frac{(\cos\alpha - 1)\sigma(-u, w) + (\cos\alpha + 1)\sigma(u, w)}{4\pi\epsilon(1-\epsilon)}$$

where

$$\omega(\alpha, \epsilon) = 4\epsilon \operatorname{Arccos}\left[\frac{1 - (\epsilon/\cos(\alpha/2))^2}{1 - \epsilon^2}\right]^{1/2} - 4 \operatorname{Arcsin} \frac{\sin(\alpha/2)}{[(1 - \epsilon^2]^{1/2}}$$

and

$$\sigma(\alpha, \epsilon) = \epsilon \operatorname{tg}\left(\frac{\alpha}{2}\right)\left[1 - \left(\frac{\epsilon}{\cos(\alpha/2)}\right)^2\right]^{1/2} - (1 - \epsilon^2) \operatorname{Arccos}\left[\frac{\epsilon \operatorname{tg}(\alpha/2)}{(1 - \epsilon^2)^{1/2}}\right]$$

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